Optimal approximation of multivariate periodic Sobolev functions

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This talk is concerned with optimal linear approximation of functions on the d-dimensional torus belonging to isotropic Sobolev spaces $H^s(\mathbb{T}^d)$ or to Sobolev spaces $H^s_{\text{mix}}(\mathbb{T}^d)$ of dominating mixed smoothness $s > 0$, where the error is measured in the L_2-norm. The asymptotic rate of decay of the corresponding approximation numbers is well known: With certain constants depending only on the dimension $d \in \mathbb{N}$ and the smoothness $s > 0$, in the isotropic case one has for all $n \in \mathbb{N}$

$$c_s(d)\, n^{-s/d} \leq a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq C_s(d)\, n^{-s/d}\,,$$

while in the mixed case it holds

$$c_s(d) \left[\frac{(\log n)^{d-1}}{n}\right]^s \leq a_n(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq C_s(d) \left[\frac{(\log n)^{d-1}}{n}\right]^s .$$

In the literature very little is known about the involved constants. However, for numerical purposes and for tractability in information-based complexity, it is essential to known not only the asymptotic rate as $n \to \infty$, but to have also information on the involved constants, especially their dependence on the dimension. In the talk I will address this question, in particular I will present new results on

- the exact asymptotic behavior of the constants as $d \to \infty$, for any fixed smoothness $s > 0$,
- rate-optimal two-sided estimates for large n,
- matching two-sided "preasymptotic" estimates, i.e. for $n \leq 2^d$, and
- a general method that allows to derive estimates for L_∞-approximation from L_2-approximation.

The proofs rely on combinatorial and volume estimates, an interesting connection to entropy numbers in finite-dimensional ℓ_p-spaces, and operator ideal techniques.

The talk is based on several recent joint papers with Fernando Cobos (Madrid), Sebastian Mayer (Bonn), Winfried Sickel (Jena) and Tino Ullrich (Bonn).